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Existence and uniqueness of solutions of a class of two-point singular nonlinear boundary value problems

M.A. El-Gebeily, A. Boumenir
Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

M.B.M. Elgindi
Department of Mathematics, Michigan State University, East Lansing, MI, United States

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Abstract

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This paper is concerned with the existence and uniqueness of solution of a class of two-point singular nonlinear boundary value problems. It is shown that the problem has a unique solution only for certain boundary conditions under the assumption that the range of $\partial f/\partial y$ has empty intersection with the closure of the spectrum of the singular differential operator, where f denotes the nonlinearity.

Keywords: Singular two-point boundary value problems; existence; uniqueness; limit circle case; limit point case; montone operators; contraction mapping theorem.

1. Introduction

We consider a differential equation of the form $\hat{l}y = f(x, y)$, where $\hat{l}$ is a second-order linear differential operator of one space variable. Such an equation is generally studied in two cases. In one case the operator $\hat{l}$ represents a regular differential expression and in the other, which is

Correspondence to: Dr. M.B.M. Elgindi, Department of Mathematics, University of Wisconsin, Eau Claire, WI 54702, United States.

more important, $\hat{l}$ represents the so-called singular expression. According to these classifications the boundary value problem

$$\begin{cases} \hat{l}(y) \equiv -\dfrac{1}{\hat{\omega}(x)}(p(x)y')' = f(x, y), \quad 0 < x < 1, \\ \lim\limits_{x \to 0^+} y(x) \cos \alpha + p(x)y'(x) \sin \alpha = 0, \\ y(1) \cos \beta + p(1)y'(1) \sin \beta = A, \end{cases} \tag{1.1}$$

with $\hat{\omega}(x)$, $p(x) \geqslant 0$, p^{-1}, $\hat{\omega} \in L^1_{\mathrm{loc}}(0, 1)$, $p^{-1} \notin L^1(0, 1)$, α, $\beta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$, A is a real constant (without loss of generality we will take $A = 0$) and $f(x, y)$ is a nonlinear forcing term, is termed *singular*. Many problems in practice [2,4,8] lead to the consideration of such singular boundary value problems. Many excellent treatments of singular linear and nonlinear boundary value problems exist in the literature. The reader is referred to [1,3,7,9] and the references therein for an extensive account of the subject.

In this paper we concentrate on two aspects of the singular boundary value problem (1.1). The first aspect we address is the behavior of the solution (if any) of (1.1) at the singular point $x = 0$. It is the authors' feeling that this point has not been treated clearly in the literature. In this respect this paper is an attempt to fill this gap. The other aspect is to apply the existing theory of nonlinear operators to the boundary value problem under consideration. In this respect we generalize the work of [5,10] on the existence and uniqueness of the solution of a special case of (1.1).

The rest of this paper is organized in four sections. Preliminary definitions and results are given in Section 2. The behavior of the solution (if any) at the singular point $x = 0$ is studied in Section 3. In Section 4 we address the nonlinear problem (1.1). Finally, the results obtained in Sections 3 and 4 are applied in Section 5 to a special class of (1.1) as an example.

2. Preliminaries

Consider the nonlinear singular boundary value problem (1.1). It is more convenient to transform the boundary value problem to one with leading coefficient unity. To do this we introduce the new independent variable t by the relation

$$t(x) = 1 + \int_x^1 \frac{1}{p(s)}\,\mathrm{d}s. \tag{2.1}$$

Under the transformation (2.1), (1.1) reduces to

$$l(y) = f(t, y), \quad 1 \leqslant t < \infty, \tag{2.2}$$

$$B_1(y) = 0, \tag{2.3}$$

$$B_\infty(y) = 0, \tag{2.4}$$

where

$$l(y) = -\frac{1}{\omega} y'', \tag{2.5}$$

$$B_1(y) = \cos \beta y(1) + \sin \beta y'(1), \tag{2.6}$$

$$B_\infty(y) = \lim_{t \to \infty} \cos \alpha y(t) + \sin \alpha y'(t), \tag{2.7}$$

$$f(t, y) = \hat{f}(x(t), y(x(t))), \tag{2.8}$$

$$\omega(t) = \hat{\omega}(x(t)) p(x(t)), \tag{2.9}$$

and the differentiations in (2.5)–(2.7) are with respect to the new variable t.

We will denote by L^2_ω the Hilbert space of all complex-valued measurable functions y which satisfy

$$\int_1^\infty |y(s)|^2 \omega(s) \, ds < \infty, \tag{2.10}$$

with inner product defined as

$$\langle y, z \rangle = \int_1^\infty y(s) \bar{z}(s) \omega(s) \, ds, \tag{2.11}$$

for y, $z \in L^2_\omega$. We also define two operators L_M and L_0 associated with the formally self-adjoint operator l by

$$D(L_M) = \{ y \in L^2_\omega \colon l(y) \in L^2_\omega \}, \tag{2.12}$$

$$L_M y = l(y), \tag{2.13}$$

and

$$D(L_0) = \{ y \in D(L_M) \colon y(1) = y'(1) = 0, \ [y, z]_\infty = 0 \text{ for all } z \in D(L_M) \}, \tag{2.14}$$

$$L_0 y = l(y), \tag{2.15}$$

where for y, $z \in D(L_M)$ we have

$$[y, z]_t = y(t)\overline{z'(t)} - y'(t)\overline{z(t)}. \tag{2.16}$$

We will assume that the nonlinearity $f(t, y)$ satisfies the condition

$$f(t, y(t)) \in L^2_\omega, \quad \text{for all } y \in L^2_\omega. \tag{2.17}$$

It is well known (see [10]) that $D(L_0)$ (and consequently $D(L_M)$) is dense in L^2_ω, that L_0 is a closed symmetric operator with $L_0^* = L_M$ and that

$$D(L_M) = D(L_0) \oplus N_\lambda \oplus N_{\bar{\lambda}}, \tag{2.18}$$

where λ is any complex number with Im $\lambda \neq 0$ and $N_\alpha = \ker(L_M - \alpha I)$. It is also known that (2.1) is a unitary transformation from $L^2_{\hat{\omega}}(0, 1]$ onto L^2_ω, which leaves the classification properties of (1.1) invariant (see [7, p.292]). In particular, whether (1.1) is in the limit circle (LC) or limit point (LP) case is left unaltered by the transformation.

Our goal in this paper is two-fold. First to determine the boundary conditions which define a self-adjoint extension of L_0. Secondly to obtain sufficient conditions on the nonlinearity $f(t, y)$ under which (2.2)–(2.4) (and consequently (1.1)) possesses a unique solution. Our first goal will be achieved by taking a closer look at the domains of L_0 and L_M and studying their properties. This task is taken up in Section 3. The nonlinear problem (2.2)–(2.4) is considered in Section 4.

3. Boundary conditions defining a self-adjoint extension of L_0

In this section we determine the boundary conditions which will define a self-adjoint extension of the minimal operator L_0. We do this by studying some properties of the domains of L_M and L_0. We shall consider both limit circle and limit point cases. The limit point case will be considered under the additional assumption that the measure of $[1, \infty)$ determined by ω is finite, i.e., $1 \in L^2_\omega$.

Limit circle case

This case is determined by the condition

$$1,\ t \in D(L_M), \tag{3.1}$$

and both are eigenfunctions of L_M corresponding to the eigenvalue 0.

For any $y \in D(L_M)$ we have

$$y'(t) = a - \int_1^t g(s)\omega(s)\ \mathrm{d}s \tag{3.2}$$

and

$$y(t) = at + b - \int_1^t (t-s)g(s)\omega(s)\ \mathrm{d}s, \tag{3.3}$$

where $g = l(y)$ and a, b are constants of integration. (Observe that $g \in L^2_\omega$ by the definition of $D(L_M)$.)

Now from (3.1), (3.2) it follows that $y' \in L^2_\omega$ and that $y'(\infty)$ is finite. Furthermore, from (3.2), (3.3) we can write

$$y(t) = b + ty'(t) + \int_1^t sg(s)\omega(s)\ \mathrm{d}s, \tag{3.4}$$

which in turn implies that a necessary (but not sufficient) condition for $y(\infty)$ to be finite is that $y'(\infty) = 0$. Now if $y'(\infty) \neq 0$, we have

$$|B_\infty(y)| = \begin{cases} \infty, & \text{if } \alpha \neq \frac{1}{2}\pi, \\ |y'(\infty)|, & \text{if } \alpha = \frac{1}{2}\pi, \end{cases} \tag{3.5}$$

from which it follows that a necessary condition for $y \in D(L_M)$ to satisfy the boundary condition (2.4) is that $y'(\infty) = 0$. This, in turn, restricts the parameter α to the value $\frac{1}{2}\pi$. Furthermore, for any $y \in D(L_M)$ with $y'(\infty) = 0$, the constant a in (3.2) is determined by

$$a = \int_1^\infty g(s)\omega(s)\ \mathrm{d}s. \tag{3.6}$$

We can now prove the following lemma.

Lemma 3.1. *The restriction* $\tilde{L}$ *of* L_M *to the space*

$$W = \{u \in D(L_M): u'(\infty) = B_1(u) = 0\}$$

is a self-adjoint extension of L_0.

Proof. For any u, $v \in D(L_M)$ we have

$$u(t) = a_1 t + b_1 - \int_1^t (t-s) g_1(s)\omega(s)\,\mathrm{d}s, \tag{3.7}$$

$$v(t) = a_2 t + b_2 - \int_1^t (t-s) g_2(s)\omega(s)\,\mathrm{d}s, \tag{3.8}$$

for some g_1, $g_2 \in L^2_\omega$ and some constants a_1, b_1, a_2, b_2. Now if u, $v \in W$ the boundary conditions $B_1(u) = B_1(v) = 0$ respectively give

$$(b_1 + a_1) \cos\beta + a_1 \sin\beta = 0, \qquad (b_2 + a_2) \cos\beta + a_2 \sin\beta = 0,$$

which implies

$$a_1 b_2 - a_2 b_1 = 0.$$

On the other hand, from (3.7), (3.8) it follows that

$$[u, v]_1 = u'(1)v(1) - u(1)v'(1) = a_1 b_2 - a_2 b_1 = 0, \tag{3.9}$$

for all u, $v \in W$.

Also for, u, $v \in W$, (3.6)–(3.8) imply

$$(u'v - uv')(t) = -\int_1^t s g_1(s)\omega(s)\,\mathrm{d}s \int_t^\infty g_2(s)\omega(s)\,\mathrm{d}s + \int_1^t s g_2(s)\omega(s)\,\mathrm{d}s \int_t^\infty g_1(s)\omega(s)\,\mathrm{d}s,$$

from which it follows that

$$[u, v]_\infty = \lim_{t\to\infty} (u'v - uv')(t) = 0. \tag{3.10}$$

It follows that any u, $v \in W$ satisfy

$$[u, v]_1 = [u, v]_\infty = 0, \tag{3.11}$$

and this proves that $\tilde{L}$ is symmetric on W.

Now since in the limit circle case the deficiency indices of L_M are (2, 2), the dimension of $D(L_M)$ modulo $D(L_0)$ is 4 and since the dimension of $D(L_M)$ modulo W is 2 (by the definition of W), we conclude that the dimension of W modulo $D(L_0)$ is also 2, and since any self-adjoint extension of L_0 has exactly codimension 2, it follows that $\tilde{L}$ is a self-adjoint extension of L_0. □

Limit point case

This case is determined by the condition

$$\int_1^\infty t^2\omega(t)\, \mathrm{d}t = \infty. \tag{3.12}$$

In addition to this condition we shall assume

$$\int_1^\infty \omega(t)\, \mathrm{d}t < \infty. \tag{3.13}$$

Under these conditions it is well known [10] that every $u, v \in D(L_M)$ satisfy

$$[u, v]_\infty = 0. \tag{3.14}$$

Now from (3.13) it follows that $1 \in D(L_M)$ and hence, by taking $v = 1$ in (3.14), each $u \in D(L_M)$ satisfies

$$u'(\infty) = 0. \tag{3.15}$$

Also in this case each $u \in D(L_M)$ can be written in the form

$$u(t) = b + \int_1^t sg(s)\omega(s)\, \mathrm{d}s + t\int_1^\infty g(s)\omega(s)\, \mathrm{d}s, \tag{3.16}$$

for some $g \in L^2_\omega$ and some constant b.

The following lemma shows that the boundary condition $B_1(u) = 0$ defines a self-adjoint extension of L_0.

Lemma 3.2. *The restriction $\tilde{L}$ of L_M to the space*

$$W = \{u \in D(L_M):\ B_1(u) = 0\}$$

is a self-adjoint extension of L_0.

Proof. Using (3.14) and an argument similar to that used in the proof of Lemma 3.1 it can be easily shown that L_M is symmetric on W. Furthermore, since in the limit point case the deficiency indices of L_M are (1, 1) and the dimension of $D(L_M)$ modulo W is 1, it follows that $\tilde{L}$ is a self-adjoint extension of L_0. □

Further properties of $u \in D(L_M)$ in the limit point case are stated in the following lemma; thus shedding more light on the nature of the solution to the nonlinear problem (2.8)–(2.10) in the limit point case. In this lemma $L^2(1, \infty)$ denotes the usual L^2-space of functions on $(1, \infty)$.

Lemma 3.3. *Every $u \in D(L_M)$ satisfies*

(i) $\lim_{t\to\infty}\bar{u}(t)u'(t) = 0$;

(ii) $u' \in L^2(1, \infty)$.

Proof. The idea of the proof is similar to that in [6, p.154]. For a real-valued $u \in D(L_M)$ we have

$$\langle L_M u, u\rangle = \lim_{t\to\infty}\left\{\int_1^t |u'|^2\, \mathrm{d}s - \bar{u}(t)u'(t)\right\} \tag{3.17}$$

$$+\bar{u}(1)u'(1), \tag{3.18}$$

from which it follows that $\lim_{t\to\infty}\bar{u}u'$ exists and is greater than $-\infty$ (it can be $+\infty$). Assume now that u is real-valued and let

$$c = \lim_{t\to\infty} uu';$$

if $c < 0$, then u^2 is decreasing for large enough t. Consequently, u is bounded far out. But then $u'(\infty) = 0$ implies that $\lim_{t\to\infty}uu' = 0$. This is a contradiction. Thus $c \geqslant 0$. If $c > 0$, then u^2 is greater than a multiple of t for large enough t. Thus $\int_1^\infty u^2\omega \, \mathrm{d}t < \infty$ implies that $\int_1^\infty t\omega \, \mathrm{d}t < \infty$. Now we have

$$\begin{aligned}
|u(t)u'(t)| &= \left|u(t_1) + \int_{t_1}^t u'(s) \, \mathrm{d}s\right| |u'(t)| \\
&\leqslant |u(t_1)u'(t)| + \left|\int_{t_1}^t \left(\int_s^\infty g(\xi)\omega(\xi) \, \mathrm{d}\xi\right) \mathrm{d}s\right| \left|\int_t^\infty g(s)\omega(s) \, \mathrm{d}s\right| \\
&\leqslant |u(t_1)u'(t)| + \|g\|_{L^2_\omega}^2 \left[\int_{t_1}^t \left(\int_s^\infty \omega(\xi) \, \mathrm{d}\xi\right)^{1/2} \mathrm{d}s\right] \left(\int_t^\infty \omega(s) \, \mathrm{d}s\right)^{1/2} \\
&\leqslant |u(t_1)u'(t)| + \|g\|_{L^2_\omega}^2 \int_{t_1}^\infty \left(\int_s^\infty \omega(\xi) \, \mathrm{d}\xi\right) \mathrm{d}s \\
&\leqslant |u(t_1)u'(t)| + \|g\|_{L^2_\omega}^2 \int_{t_1}^\infty s\omega(s) \, \mathrm{d}s,
\end{aligned}$$

from which it follows, by taking t_1 large enough and using $u'(\infty) = 0$, that $|uu'| \to 0$ as $t \to \infty$, which is a contradiction. This completes the proof of (i) for a real-valued $u \in D(L_M)$. For a complex-valued $u \in D(L_M)$ we can write $u = u_1 + \mathrm{i}u_2$. Since u, $\bar{u} \in D(L_M)$, it follows that u_1, $u_2 \in D(L_M)$. Thus

$$\lim_{t\to\infty} u_1u_1' = \lim_{t\to\infty} u_2u_2' = 0. \tag{3.19}$$

From (3.19) it follows that

$$\lim_{t\to\infty} \bar{u}u' = 2\mathrm{i} \lim_{t\to\infty} u_1u_2'. \tag{3.20}$$

Since the limit in (3.18) exists and is finite, we have by (3.20),

$$\left(\lim_{t\to\infty} \bar{u}u'\right)^2 = -4 \lim_{t\to\infty} (u_1u_2')^2 \tag{3.21}$$

$$= -4 \lim_{t\to\infty} u_1u_2'u_1'u_2 = 0. \tag{3.22}$$

This completes the proof of (i).

Part (ii) of the lemma is an immediate consequence of part (i) and (3.18). □

Example 3.4. A solution of the differential equation

$$-t^2y''(t) = -\tfrac{3}{16}t^{1/4}, \quad t \in (1, \infty),$$

is

$$y(t) = t^{1/4} + c,$$

where c is a constant of integration. Observe that here $\omega(t) = t^{-2}$, which satisfies the conditions (3.12) and (3.13) of the limit point case considered above. This example shows that the results of Lemma 3.3 cannot, in general, be improved upon.

4. The nonlinear problem

We now turn to the study of existence and uniqueness of the solution to the boundary value problem (2.2)–(2.4) (or equivalently (1.1)) with $\alpha = \frac{1}{2}\pi$, $\beta \in (\frac{1}{2}\pi, \frac{1}{2}\pi]$.

In what follows, $\tilde{L}$ will denote the self-adjoint extension defined by Lemma 3.1 or Lemma 3.2 depending upon whether the problem is in the limit circle case or the limit point case with condition (3.13) being satisfied. With this, (2.2)–(2.4) can be written in the form

$$\tilde{L}y = f(t, y), \quad y \in W, \tag{4.1}$$

where $t \in (1, \infty)$ and W is defined as in Lemma 3.1 or Lemma 3.2 depending upon whether ω satisfies condition (3.1) or conditions (3.12) and (3.13) respectively. We shall prove that (4.1) has a unique solution under the assumption

$$\left[\mathrm{Inf}\frac{\partial f}{\partial y}, \mathrm{Sup}\frac{\partial f}{\partial y}\right] \cap \overline{\sigma(\tilde{L})} = \emptyset, \tag{4.2}$$

where $\mathrm{Inf}(\partial f/\partial y)$ and $\mathrm{Sup}(\partial f/\partial y)$ are taken over $\{(t, y)\colon 1 \leqslant t < \infty, -\infty < y < \infty\}$ and $\overline{\sigma(\tilde{L})}$ denotes the closure of the spectrum of $\tilde{L}$. Our analysis will then be carried in two situations:

$$(1) \qquad -\infty \leqslant \mathrm{Inf}\frac{\partial f}{\partial y} \leqslant \mathrm{Sup}\frac{\partial f}{\partial y} < K_1, \tag{4.3}$$

where $K_1 = \mathrm{Inf}\{\lambda \in \mathbb{R}\colon \lambda \in \sigma(\tilde{L})\}$;

$$(2) \qquad K_2 < \mathrm{Inf}\frac{\partial f}{\partial y} \leqslant \mathrm{Sup}\frac{\partial f}{\partial y} < K_3, \tag{4.4}$$

where $K_2 = \mathrm{Sup}\{\lambda \in \mathbb{R}\colon \lambda \in \sigma(\tilde{L}), \lambda < \mathrm{Inf}(\partial f/\partial y)\}$, $K_3 = \mathrm{Inf}\{\lambda \in \mathbb{R}\colon \lambda \in \sigma(\tilde{L}), \lambda > \mathrm{Sup}(\partial f/\partial y)\}$.

We emphasize that no assumption of discreteness of $\sigma(\tilde{L})$ or of boundedness of $f(t, y)$ has been made throughout this work.

We now state and prove two theorems corresponding to the two situations (1) and (2) above.

Theorem 4.1. *If* (4.2) *and* (4.3) *hold, then* (4.1) *has a unique solution.*

Proof. It is readily checked that

(i) $K_1 - f(\cdot, y)$ is a monotone hemicontinuous operator from L^2_ω into itself;
(ii) $\tilde{L} - K_1$ is a maximal monotone operator from W onto L^2_ω;
(iii) $\tilde{L} - f(\cdot, y)$ is coercive, i.e., $\|(\tilde{L} - f)y\|_{L^2_\omega} \to \infty$ as $\|y\|_{L^2_\omega} \to \infty$.

It follows from [3, corollary on p.48] that $\tilde{L} - f$ is onto. This proves existence of solution in this case. Uniqueness follows from

$$\langle Ty - Tz, y - z\rangle \geqslant (K_1 - d)\langle y - z, y - z\rangle,$$

where $T = \tilde{L} - f$, $d = \operatorname{Sup}(\partial f/\partial y)$ and y, $z \in D(\tilde{L})$. This completes the proof of the theorem. □

Theorem 4.2. *If* (4.2) *and* (4.4) *hold, then* (4.1) *has a unique solution.*

Proof. The proof is based upon the Contraction Mapping Theorem and is therefore constructive. We define

$$c = \operatorname{Inf}\frac{\partial f}{\partial y}, \qquad d = \operatorname{Sup}\frac{\partial f}{\partial y}.$$

Take any $k \in [c, d]$. Then $k \notin \sigma(\tilde{L})$. We seek to choose k so that $(\tilde{L} - k)^{-1}(f - k)$ is a contraction. First we observe that

$$\|(\tilde{L} - k)^{-1}\| = [\operatorname{Min}\{(k - K_2), (K_3 - k)\}]^{-1} := \delta(k),$$

and for any y, $z \in L^2_\omega$,

$$\|(\tilde{L} - k)^{-1}[(f - k)y - (f - k)z]\| \leqslant \delta(k) \operatorname{Max}\{|d - k|, |c - k|\} \|y - z\|.$$

Thus we need to find k so that

$$q(k) := \delta(k) \operatorname{Max}\{|d - k|, |c - k|\} < 1.$$

It can be shown that $q(k)$ assumes its minimum value

$$q^* = \frac{d - c}{(K_3 - K_2) - |K_3 + K_2 - c - d|},$$

when $k = \frac{1}{2}(c + d)$ and that $q^* < 1$. With this choice of k, $(\tilde{L} - k)^{-1}(f - k)$ is a contraction and it has a unique fixed point y^* which is a solution of (4.1). □

We observe that condition (4.4) assumes the sets

$$S_1 =: \sigma(\tilde{L}) \cap (-\infty, c) \quad \text{and} \quad S_2 =: \sigma(\tilde{L}) \cap (d, \infty)$$

to be nonempty. The cases when $S_1 = \emptyset$ or $S_2 = \emptyset$ are treated as follows. In the first case, K_2 is undefined and because of the assumption (4.2) we have $\sigma(\tilde{L}) \subseteq (d, \infty)$, $K_3 = K_1$ and $d < K_1$. The proof of the existence and uniqueness of the solution of (4.1) in this case follows from Theorem 4.1. In the second case, K_3 is undefined and because of the assumption (4.2) we have $\sigma(\tilde{L}) \subseteq (-\infty, c)$ and $K_2 < c$. In this case the proof of the existence and uniqueness of the solution of (4.1) follows by applying the Contraction Mapping Theorem to the equation

$$(\tilde{L} - k)y = f(x, y) - ky,$$

where $k = \frac{1}{2}(c + d)$, as in the proof of Theorem 4.2. These observations complete the proof of the main theorem of this section.

Theorem 4.3. *Equation* (4.1) *has a unique solution provided that the closure of* $\sigma(\tilde{L})$ *does not intersect* $[\operatorname{Inf}(\partial f/\partial y), \operatorname{Sup}(\partial f/\partial y)]$.

Remarks. (1) The choice of $k = \frac{1}{2}(c+d)$ in the proof of Theorem 4.2 is optimal in the sense that it minimizes the norm of the contraction operator, thus increasing the rate of convergence of Picard's iterations.

(2) Our results in this paper generalize those obtained recently [5] in several directions.

(3) Numerical methods for solving singular boundary value problems of the form considered here are currently in progress by the authors.

5. Example

In this section we apply the results of the previous sections to the boundary value problem

$$\begin{cases} -\dfrac{1}{x^\gamma}(x^\alpha y')' = f(x, y), & 0 < x < 1, \\ \lim\limits_{x \to 0^+} x^\alpha y'(x) = 0, & y(1)\cos\beta + y'(1)\sin\beta = A, \end{cases} \tag{5.1}$$

where $\alpha \geqslant 1$, $\gamma > 0$, $\beta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and A is a real constant.

The analysis of existence and uniqueness of the solution of a special case of (5.1) has been considered in [5], in which the authors assumed $\alpha = \gamma$ and $f(x, y)$ is bounded.

Since the eigenvalues of $\tilde{L}$ in this case are the positive zeros of the Bessel function

$$J_{(\alpha-1)/(\gamma+2-\alpha)}(\sqrt{\lambda}) = 0, \tag{5.2}$$

the conditions on $f(x, y)$ stated in Section 4 are equivalent to requiring that either $\operatorname{Sup}(\partial f/\partial y)$ be less than the first zero of (5.2) or the interval $[\operatorname{Inf}(\partial f/\partial y), \operatorname{Sup}(\partial f/\partial y)]$ lies between any two consecutive zeros of (5.2). Satisfaction of any of these two conditions guarantees the existence and uniqueness of a solution of (5.1) by Theorems 4.1 and 4.2. In this example we have

$$\omega(t) = \begin{cases} [2-\alpha+(\alpha-1)t]^{(\gamma+\alpha)/(1-\alpha)}, & \alpha > 1, \\ e^{(\alpha+\gamma)(1-t)}, & \alpha = 1, \end{cases}$$

from which it follows that $\int_1^\infty \omega(t)\,dt < \infty$ for all $\alpha \geqslant 1$, $\gamma > 0$ and that (5.1) is in the limit circle case if $2\alpha \leqslant \gamma + 3$ and is in the limit point case otherwise. In the special case of [5] ($\alpha = \gamma$) the limit circle case corresponds to $\alpha \leqslant 3$ and the limit point case corresponds to $\alpha > 3$. Finally we remark that the results of [5] will follow from the following lemma.

Lemma 5.1. *If $\gamma > \alpha - 1$ and $f(x, y)$ is bounded and satisfies either of the conditions* (4.3) *or* (4.4), *then the unique solution $y(x)$ of* (5.1) *satisfies*

$$\lim_{x \to 0^+} y'(x) = 0. \tag{5.3}$$

Proof. Since $y(x)$ satisfies

$$(x^\alpha y'(x))' = -x^\gamma f(x, y(x)),$$

it follows that

$$y'(x) = \frac{\int_0^x t^\gamma f(t, y(t))\,\mathrm{d}t}{x^\alpha},$$

and hence

$$\lim_{x \to 0^+} y'(x) = \lim_{x \to 0} x^{\gamma-\alpha+1} f(x, y(x)),$$

which is equal to zero since $\gamma > \alpha - 1$ and $f(x, y)$ is bounded. $\square$

We should note here in passing that the solution of (5.1) does not in general satisfy (5.3). For example, the equation $-(xy')' = -2 - \mathrm{Ln}\, x$ has the solution $y(x) = x\,\mathrm{Ln}\, x$ subject to the boundary conditions $\lim_{x \to 0^+} xy'(x) = 0$ and $y(1) = 0$. However, $\lim_{x \to 0^+} y'(x) \neq 0$.

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